Double Integrals w/ spheres

Last time: Double Integrals

This time: Double Integrals

Q: Suppose a sphere has radius α and is centered at the origin. What is its volume?

$x^2+y^2+z^2=\alpha^2$

Sol: we seek volume via double integral

For height function, $z^2=\alpha^2-x^2-y^2$

so upper hemisphere has height $z=\sqrt{\alpha^2-x^2-y^2}$

lower hemisphere has height $z=-\sqrt{\alpha^2-x^2-y^2}$

$\therefore$ Total height is "(upper hemisphere) - (lower hemisphere)"

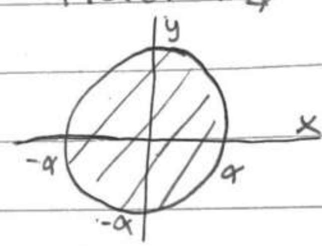
i.e. $h(x,y)=\sqrt{\alpha^2-x^2-y^2}-\left(-\sqrt{\alpha^2-x^2-y^2}\right)$

$=2\sqrt{\alpha^2-x^2-y^2}$

To parameterize R_α, we note $R_\alpha=\{(x,y): x^2+y^2\le\alpha^2\}$

$=\{(x,y): -\alpha\le x\le\alpha,\ -\sqrt{\alpha^2-x^2}\le y\le\sqrt{\alpha^2-x^2}\}$

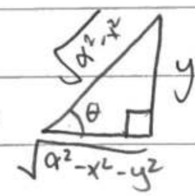
Pic. of R_α



$\therefore \mathrm{Vol}(S)=\iint_R h(x,y)\,dA$

$=\int_{x=-\alpha}^{\alpha}\int_{y=-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}} 2\sqrt{\alpha^2-x^2-y^2}\,dy\,dx$

Inner Intr. $\int 2\sqrt{(\alpha^2-x^2)-y^2}\,dy \longrightarrow$



$=2\int\sqrt{\alpha^2-x^2}\cos\theta\sqrt{\alpha^2-x^2}\cos\theta\,d\theta$

$y=\sqrt{\alpha^2-x^2}\sin\theta$

$dy=\sqrt{\alpha^2-x^2}\cos\theta\,d\theta$

$=2(\alpha^2-x^2)\int\cos^2(\theta)\,d\theta$

$=(\alpha^2-x^2)\int(1+\cos(2\theta))\,d\theta$

$\sin(2\theta)=2\sin\theta\cos\theta$

$=(\alpha^2-x^2)\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C$

$=(\alpha^2-x^2)(\theta+\sin\theta\cos\theta)+C$

$=(\alpha^2-x^2)\left(\arcsin\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)+\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)\cdot\left(\frac{\sqrt{\alpha^2-x^2-y^2}}{\sqrt{\alpha^2-x^2}}\right)\right)+C$

$=(\alpha^2-x^2)\left(\arcsin\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)\right)+y\sqrt{\alpha^2-x^2-y^2}+C$

$\longrightarrow$

Sol: →

$\therefore$ evaluating we obtain

$$\int_{y=\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}} 2\sqrt{\alpha^2-x^2-y^2}\,dy = \left[(\alpha^2-x^2)\arcsin\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)+y\sqrt{\alpha^2-x^2-y^2}\right]_{y=\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-y^2}}$$

$$=\left((\alpha^2-x^2)\arcsin(1)+\sqrt{\alpha^2-x^2}\sqrt{0}\right)-\left((\alpha^2-x^2)\arcsin(-1)-\sqrt{\alpha^2-x^2}\sqrt{0}\right)$$

$$=(\alpha^2-x^2)\left(\arcsin(1)-\arcsin(-1)\right)=(\alpha^2-x^2)\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)$$

$$=\pi(\alpha^2-x^2)$$

Outer Integral: $\displaystyle\int_{x=-\alpha}^{\alpha}\pi(\alpha^2-x^2)=\pi\int_{x=-\alpha}^{\alpha}(\alpha^2-x^2)\,dx$

$$=\pi\left[\alpha^2x-\frac{1}{3}x^3\right]_{x=-\alpha}^{\alpha}=\pi\left(\left(\alpha^3-\frac{1}{3}\alpha^3\right)-\left(-\alpha^3+\frac{1}{3}\alpha^3\right)\right)$$

$$=\pi\left(2\alpha^3-\frac{2}{3}\alpha^3\right)=\frac{4}{3}\pi\alpha^3$$

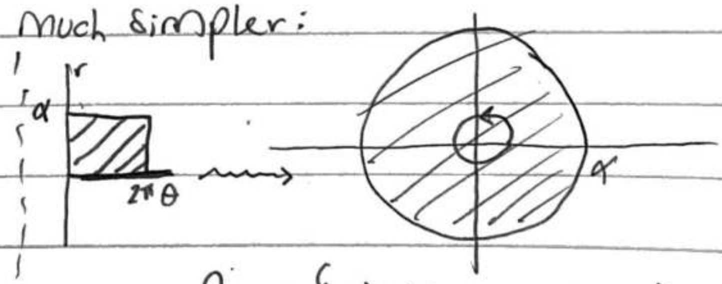
Hence the $V(S_\alpha)=\frac{4}{3}\pi\alpha^3$ is the volume of a sphere of radius $\alpha>0$ $\square$

NB: That was Computationally Complicated. If we understand. how to use polar coordinates for integration, both the radius and height function are much simpler:

height function $h(x,y)=\sqrt{\alpha^2-x^2-y^2}$

So, $h(r\cos\theta, r\sin\theta)=\sqrt[2]{\alpha^2-r^2}$

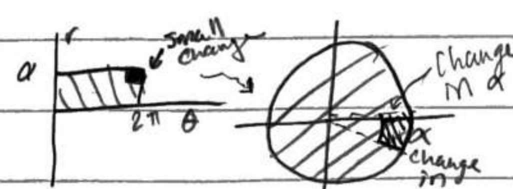
NEED: Understand dA polar and how it relates to dA cartesian.



$R_{polar}=\{(r,\theta): 0\le r\le\alpha \quad 0\le\theta\le 2\pi\}$

Rectangle in Polar plane

To understand dApolar, Consider a small



polar rectangle. In the Cartesian plane this corresponds to a Circular sector.

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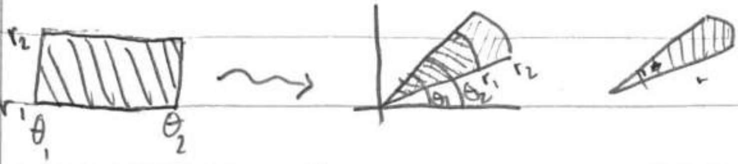


$\frac{1}{2}r^2\theta$ = area of a full circular sheet

Area of a (small) circular sector is:

$$\left(\frac{1}{2}r_2^2(\theta_2-\theta_1)\right)-\left(\frac{1}{2}r_1^2(\theta_2-\theta_1)\right)=\frac{1}{2}(\theta_2-\theta_1)(r_2^2-r_1^2)=\frac{1}{2}(r_2+r_1)(r_2+r_1)(\theta_2-\theta_1)$$

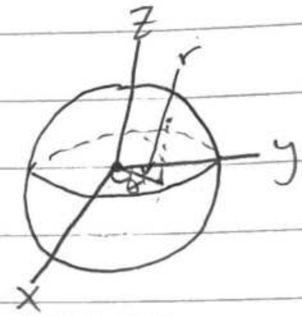
$$\therefore \Delta A_{cart} = \frac{1}{2}(r_1+r_2)\Delta r\,\Delta\theta = \frac{1}{2}(r_1+r_2)\,\Delta A_{polar}$$

Now limiting as $\Delta A \to 0$ (i.e. $\Delta\theta \to 0$ and $\Delta r \to 0$).

we see $\frac{1}{2}(r_1+r_2) \longrightarrow \frac{1}{2}2r^* = r^*$

Hence in the limit we obtain $\underset{\substack{dy\,dx \\ \text{or} \\ dx\,dy}}{dA_{cart}} = r\,\underset{\substack{dr\,d\theta \\ \text{or} \\ d\theta\,dr}}{dA_{polar}}$

Volume of Sphere, a. (polar coordinates)



Sol: In polar Coords (r,θ,z) space, Integral becomes

$$Vol(S_a) = \iint_{R_{cart}} h(x,y)\,dA_{cart}$$

$$= \iint_{R_{polar}} h(r\cos\theta, r\sin\theta)\,r\,dA_{polar} = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} 2\sqrt{a^2-r^2}\;r\,dr\,d\theta$$

$$Z=\sqrt{a^2-x^2-y^2} = \sqrt{a^2-r^2}$$

$$R_{polar} = \underset{(\theta,r)}{[0,2\pi]\times[0,a]}$$

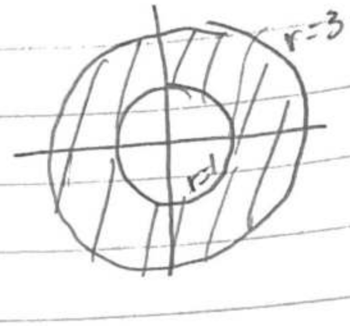
Inner Integral: $\int_{r=0}^{a} 2r\sqrt{a-r^2}\,dr$ $\quad v=a^2-r^2$, $dv=-2r\,dr$

$$= -\int_{r=0}^{a} v^{1/2}\,dv = -\left[\frac{2}{3}v^{3/2}\right]_{r=0}^{a}$$

$$= -\frac{2}{3}\left[(a^2-r^2)^{3/2}\right]_{r=0}^{a} = \frac{-2}{3}\left((0)^{3/2}-(a^2)^{3/2}\right) = \frac{-2}{3}(-a^3) = \frac{2}{3}a^3$$

Outer: $\int_{\theta=0}^{2\pi}\frac{2}{3}a^3\,d\theta = \frac{2}{3}\left[a^3\theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3 \quad \square$

Ex: Compute $\iint_R \sin(\sqrt{x^2+y^2})\,dA$ for R the annulus between the circles $x^2+y^2=1$ and $x^2+y^2=9$.



Sol: Turn the integral polar

$R_{polar} = \{(r,\theta): 0 \le \theta \le 2\pi,\ 1 \le r \le 3\}$

$f(x,y) = \sin(\sqrt{x^2+y^2})$ has polar form

$f(r\cos\theta, r\sin\theta) = \sin(\sqrt{r^2}) = \sin(r)$

$\therefore \iint_{R_{cart}} \sin(\sqrt{x^2+y^2})\,dA_{cart} = \iint_{R_{pol}} \sin(r)\, r\, dA_{pol}$

$= \int_{\theta=0}^{2\pi} \int_{r=1}^{3} \sin(r) r \, dr\, d\theta \quad \begin{pmatrix} u=r & dv=\sin(r)dr \\ du=dr & v=-\cos(r) \end{pmatrix}$

$= \int_{\theta=0}^{2\pi} \left[-r\cos(r) - \int -\cos(r)\,dr\right]_{r=1}^{3} d\theta$

$= \int_{\theta=0}^{2\pi} \left[-r\cos(r) + \sin(r)\right]_{r=1}^{3} d\theta = \int_{\theta=0}^{2\pi} ((-3\cos(3)+\sin(3)) - (-1\cos(1)+\sin(1)))$

$= \underline{(\sin(3) - \sin(1) - 3\cos(3) + \cos(1))} [\theta]_{\theta=0}^{2\pi} = 2\pi(\text{''} \quad \text{''})$

Exercise: compute $\iint_R x\exp(-x^2-y^2)\,dA$ on R the disk of radius R about the origin